Anatomy of a Jerk: A Dissertation on the Effects of the Third Derivative on Graphs in the Two Dimensional Cartesian Plane

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Abstract

While the numerical properties of the third derivative, known in physics as the jerk, as well as its practical uses, have been well established, very little literature exists to describe the visual properties of the original function impacted by its third derivative. To attempt to answer this, 25 equations, split into groups of 5, were quasi-randomly selected and graphed, along with their third derivatives, then analyzed to find correlations. These correlations were then logically explained in an attempt to filter out coincidences. The conclusion drawn was that the changing concavity caused by the third derivative would create an “s” shape in the graph, with the sign of the third derivative describing which direction the “s” shape took and the absolute value of the third derivative dictating how compressed or elongated the “s” shape is.

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One of the fundamental concepts taught in introductory Calculus is how the first and second derivatives affect the graph of the original function. If the first derivative is positive, the graph increases, and vice versa. The second derivative impacts concavity. However, the third derivative – the natural succession – is never fully addressed nor explained. While a wide body of literature has focused on practical applications of the third derivative and how it can be used in various equations, information on how the third derivative itself impacts the shape of the function is scant to non-existent. This paper aims to address this by systematically examining various graphs and then attempting to logically prove any correlations found.

Background

The majority of existing literature focused on four major concepts in relation to the third derivative, each of which will be addressed in detail. The most obvious of the four was the concept of jerk. Used in physics, jerk is the term from which both the third derivative and this paper derives their names. Relatively well-defined, the jerk of a function has many practical applications. The second concept was when the third derivative of a curve in three-dimensional space is defined as torsion. While not specific to the analysis of two-dimensional graphs, the information was still helpful. The third definition examined was one of penosculating conics. In brief, they are conics related to a curve that the third derivative is used to help find. The fourth and final definition used was the one of aberrancy, which describes how “circle-like” a curve is at various points.

Jerk

While the exact origin of the word Jerk was unclear, its applications in physics certainly were not. Just as velocity was the derivative of position and acceleration was the derivative of velocity, the third derivative was the derivative, or rate of change, of acceleration. The technical definition was typically given as "the derivative of acceleration with respect to time ($\frac{da}{dt}$)." Because the third derivative described when acceleration changes, it was very useful in finding large changes in acceleration over short durations. To humans inside a moving vehicle, these sudden accelerations would have been felt as a jerk, which, scientists believed, was where the third derivative most likely received its name. Calculating (and, by extension, minimizing) jerk had many useful applications in the field of physics, specifically in reducing unnecessary strain on infrastructure and pain or discomfort experienced by humans during transport. Specific examples included designing railways and highways (creating arcs that caused less discomfort and strain when turning at relatively high speeds) and generating flow noise in acoustics (Schot, 1978b).

Torsion

An osculating plane was a plane defined by the three points:

$$\lim_{c,h\to 0}\big(x, f(x)\big), \big(x + c, f(x + c)\big), (x + h, f(x + h))$$

where $f(x)$ was defined as the original curve (Weisstein, n.d. a). As evident, an osculating plane was a plane that touches a three-dimensional curve at exactly one point. At this point, it was noted that the definition of a tangent line is was a line defined by two points:

$$\lim_{c\to 0}\big(x, f(x)\big), (x + c, f(x + c))$$

where $f(x)$ was defined as the original line. Therefore, conceptually, an osculating plane was to a curve in the third dimension what a tangent line was to a line in the second dimension.

Torsion was defined as the rate of change of a curve's osculating plane. The torsion of a curve was positive if the curve was right-handed and was negative if the curve was left-handed (Weisstein, n.d. b). Unfortunately, further analysis of the torsion was beyond the scope of this paper and was also unhelpful, as torsion only applied to curves in the third dimension.

Penosculating Conics

A curve's osculating circle is the circle whose first and second derivative are identical to those of a curve at a certain point along it (commonly known as second-order contact). Continuing the trend, there existed an osculant whose first, second, and third derivatives are identical at a certain point, known as an osculating parabola. In other words, the osculating parabola was the parabola with third-order contact to a curve. The axis of the osculating parabola formed an angle with the normal line (the line perpendicular to the tangent line) at a point along the curve – this was called the angle of aberrancy, a topic covered in more detail during the next section. Using the angle of aberrancy, the diameter of all osculating conics with third-degree contact could be found, resulting in a pencil of conics. Out of these, only one also shared a fourth derivative with the original curve and thus only one had fourth-degree contact. Since the pencil of conics was only one parameter short of defining a single curve, the pencil of conics that form around the curve at that point has been termed the pencil of penosculating conics (Schot, 1979c).

This pencil was useful in defining aberrancy, a fundamental component of the third derivative.

Aberrancy

While, as discussed earlier, aberrancy could be defined in terms of penosculating conics, conceptually easier definitions existed and were preferable. To find aberrancy, a chord parallel to the tangent line at $f(c)$ that intersects $f(x)$ twice, once on each side of $f(c)$, was found. The midpoint of the chord was called (u, v), and the distance between (u, v) and $(c, f(c))$ was called z. Clearly, as z approaches 0, (u, v) approaches $(c, f(c))$. The axis of aberrancy was the line formed by (u, v) and $(c, f(c))$ as z approaches 0. The angle between the axis of aberrancy and the normal line was called the angle of aberrancy, and the aberrancy of f at c was defined as the tangent of the angle of aberrancy.

As evident, the aberrancy of a curve served as an easy way to gauge how "circle-like" a curve was. A perfect circle had tangent lines perpendicular to all its radii, and thus, any chord that was parallel to the tangent line must have also been perpendicular to the radius. Since a circle curved away from a point equally on both sides, the midpoint of any such chord must have formed a line perpendicular to the tangent line, thus making it equal to the radius / normal line and creating an angle of aberrancy of 0, resulting in an aberrancy of 0. Any change in aberrancy represents a deviation in the curve from a perfect circle. For instance, a logarithmic curve has a constant angle of aberrancy around, as the deviation from a circle from one point to the next remains constant. Furthermore, a line with no curve has an undefined aberrancy (Byerley & Gordon, 2006).

Further calculation revealed that the aberrancy was given by the following formula:

$$A = f'(x) - \frac{1 + f'(x)^2}{3 * f''(x)^2} * f'''(x)$$

which bore striking resemblance to the formula used to calculate curvature, commonly used when referring to concavity:

$$k = \frac{f''(x)}{(1 + f'(x)^2)^{\frac{3}{2}}}$$

derived from the 2nd derivative (Schot, 1978a). From here, it became clear that aberrancy was a fundamental property of the third derivative the same way concavity and curvature was a fundamental property of the second derivative. Unfortunately, what this did not provide was a way to easily and visually identify properties of the third derivative in a way similar to slope and the first derivative or concavity and the second derivative. This is where this paper's analysis began.

Question

What effect does the sign and absolute value of the third derivative of a function have upon the graph of the original function in a two-dimensional Cartesian plane?

Methodology

In order to find the effects of the third derivative, I adopted a two-step process: empirics gathering followed by logical explanations.

The first step was one of empirics. I created five groups of similar equations: cubics, quartics, quintics, special equations, and edge cases. Cubics were filled with random polynomials of degree 3, quartics with degree 4, and quintics with degree 5. Special equations were filled with famous, commonly used equations, and edge cases were filled with a random smattering of additional equations. The categories were chosen quasi-randomly with the express intent of gathering a representative sample of all equations where a third derivative would be applicable. In addition, all equations within the categories were also chosen quasi-randomly with the express intent of gathering a representative sample of equations in each specific category. Though a through listing of equations used can be found in the Appendix A, the vagueness used in describing the selection process is intentional, as these results should be reproducible with any collection of equations. These equations, along with their third derivatives, were graphed with default settings, unless otherwise noted in Appendix B, with the exception of resolution, which was set significantly higher for legibility, via Wolfram Mathematica. Through post-processing, areas of the graph where the third derivative was positive were shaded green while areas where the third derivative was negative were shaded red (if viewed in black and white, red shows up as dark grey and green shows up as light grey). These graphs were reproduced in Appendix B (where the blue line represents the original function and the orange line represents the third derivative). These graphs were then analyzed by hand to determine similarities and correlations between the third derivative and the behavior of the graphs.

The second step was one of logical analysis. Using information gathered in step one, along with the information retrieved during research, I attempted to logically and mathematically explain any correlations I found. This allowed me to assert, with a reasonable degree of confidence, that any conclusions drawn would apply to all equations and were not just a fluke as a result of coincidence.

I would note that various limitations were in place. Specifically, I was limited both in terms of time and my own knowledge. With only one week to complete the research, I was only able to test the 25 equations detailed in Appendix A. While the second step of my research process was designed so that this was not a significant hindrance in terms of result accuracy, logical analysis is still no true substitute for having ample empirics. In addition, having only completed the equivalent of Calculus 2, my research was severely limited by the scope of my own knowledge. This became a problem because a significant portion of my time had to be spent understanding the existing literature, most of which was written for a higher level math than I had taken, instead of potentially conducting more trials.

Results

Upon examination of the graphs, the immediate first obvious feature was from Figure 18, the graph of $y = \sin(x)$. A clear, distinct shape in the form of the letter “s” was present in all areas where the third derivative was negative, while the same shape reversed (similar to the numeral "2") was present in all areas where the third derivative was positive. Further examination revealed that the same "s" shape was present, though sideways, in Figure 3, where the 3rd derivative stays negative for the entirety of the graph, and that the same reversed "s" shape was present in Figures 1 and 2, where the 3rd derivatives stay positive for the entirety of the graphs.

The "s" shape was most prominent when composed of a period of negative slope, followed by a prolonged period of positive slope, followed by a period of negative slope. The reversed "s" shape was most prominent when composed of a period of positive slope, followed by a prolonged period of negative slope, followed by a period of positive slope. However, in the interest of accuracy, a more precise definition was desirable. Careful consideration of the above definitions revealed that the "s" shape was a result of a period of negative concavity followed by a period of positive concavity, and that the reverse "s" shape was a result of a period of positive concavity followed by a period of negative concavity. As a result, for the remainder of this paper, I defined "s" shape as a period of increasing concavity while reverse "s" shape was defined as a period of decreasing concavity.

A wider examination of all graphs provided in Appendix B revealed that all areas with a negative 3rd derivative either directly displayed or contained parts of the prominent "s" shape, while the areas with positive 3rd derivatives displayed or contained parts of the reverse "s" shape. As expected, areas with a 3rd derivative of 0 or an undefined 3rd derivative contained no such figures.

With this strong correlation established, I logically examined the concept of the first two derivatives. The first derivative, as is common knowledge, described the change in position of the original function, known as slope. The second derivative described the change in slope of the original function, known as concavity. Thus, by logical extension, the third derivative should have described the change in concavity of the original function. I returned to my original definitions of “s” shape and reverse “s” shape. I had originally defined “s” shape as a period of decreasing concavity. By defining the third derivative as the change in concavity, I reworded the definition of “s” shape as a period where the third derivative was negative, and, by similar logic, I reworded the definition of reverse “s” shape as a period where the third derivative was positive. Given that these definitions correspond precisely to the observations I detailed above, I asserted, with reasonable degrees of confidence, that the sign of the third derivative visually impacted the direction of the “s” shape of the original graph.

With the first half of the question answered, I focused my attentions on the second portion, which asked how the absolute value of the third derivative impacted the graph of the original function. The first graph that was thoroughly examined was Figure 15, in which I noted that, across the left portion of the graph, the “s” shape seemed to get more horizontally elongated the closer to 0 the third derivative got. The same trend was noticeable in Figure 14, where the further right I looked, the steeper the slope and the sharper the bends. After confirming the observations in the other graphs, I concluded that the absolute value of the third derivative impacted how elongated the “s” curve was.

This, when examined logically, made sense because as the third derivative gets further from 0, the second derivative changes at a faster rate, forcing any changes in concavity in the original function to happen at higher speeds, thus compressing the “s” shape.

Another logical explanation came from examining the behavior of the second derivative. While the numerical consequence of the second derivative was significant at various extremes, visually, it made little difference. The most noticeable portion of the second derivative was how quickly it forces the slope to change direction at the local extrema. This was the same logic that applies to the third derivative and how it stretches the "s" shape, which is where the changes in concavity take place, instead of visually impacting the extremes.

Conclusions

The more layers of abstraction exist between a derivative and a graph, the harder it becomes to conceptualize. It was my hope that the information presented in this paper afforded a greater understanding of the third derivative. With that said, there are certain topics that I would have looked into if various limitations were not in place.

To begin, as stated earlier, I would have liked to have the time to conduct more trials, as even though I'm fairly certain of the accuracy of my results, more data is never unhelpful.

Second, similar to how the zeros of the first derivative created local extrema and how the zeros of the second derivative created points of inflection, I would have liked to examine the effects of the third derivative when it crosses or touches 0. Unfortunately, in order to do that, a more thorough investigation of the second derivative would have been necessary. Specifically, in order to address what happens when the third derivative hits zero, I must understand how the absolute value of the second derivative affects the graph both visually and numerically as a result of curvature – a concept beyond both the scope of this paper and my own knowledge.

Finally, a more thorough analysis of practical applications of the third derivative would have been desirable. Though the concept of jerk and its applications in real-world physics was touched on in the course of this paper, I would have preferred more specific examples as to how knowing the third derivative would impact life.

Still, I feel the information gained from this experience was helpful. Specifically, in the process of finding the third derivative, a pattern was established that should hopefully be equally applicable to finding the fourth and higher order derivatives, allowing for continued investigation. While the numerical properties of derivatives have been well established, my hope is that the information presented here serves as a base for further research into visually identifying derivatives quickly and easily, saving time on future calculations.

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Appendix A

Table A1

Polynomial Equations Tested

Cubics	Quartics	Quintics
$y = x^3$	$y = x^4$	$y = x^5$
$y = 2x^3 + 1$	$y = \frac{3}{4}x^4$	$y = -3x^5 - x^4$
$y = -2x^3 + 5x^2$	$y = -2x^4 + 3x^3$	$y = 2x^5 + x^4 - x^3$
$y = 2x^3 - 3x^2 + x$	$y = 2x^4 - \frac{3}{2}x^3 + 3x^2$	$y = x^5 - 3x^2 + \frac{1}{2}x + 1$
$y = -3x^3 - 2x^2 - x + 2$	$y = x^4 + 2x^3 + 5x - 1$	$y = -x^5 - 2x^4 + 3x^3 + \frac{1}{2}x^2 - 6x$

Note: These are the quasi-randomly selected polynomials tested throughout the paper. They represent the first three of five groups of equations tested. The results found should be reproducible with any equation selected, as long as it contains a third derivative.

Table A2

Other Equations Tested

Special Equations	Edge Cases
$y = e^x$	$y = x^2$
$y = \ln(x)$	$y = \sqrt{x}$
$y = \frac{1}{x}$	$y = \arcsin(x)$
$y = \sin(x)$	$y = 2^{\frac{1}{x}}$
$y = \tan(x)$	$y = 7x^7 - \frac{8}{3}x^3 - 3x^2 + x$

Note: These are the remaining quasi-randomly selected equations tested throughout the paper. They represent the last two of five groups of equations tested. The results found should be reproducible with any equation selected.

Appendix B

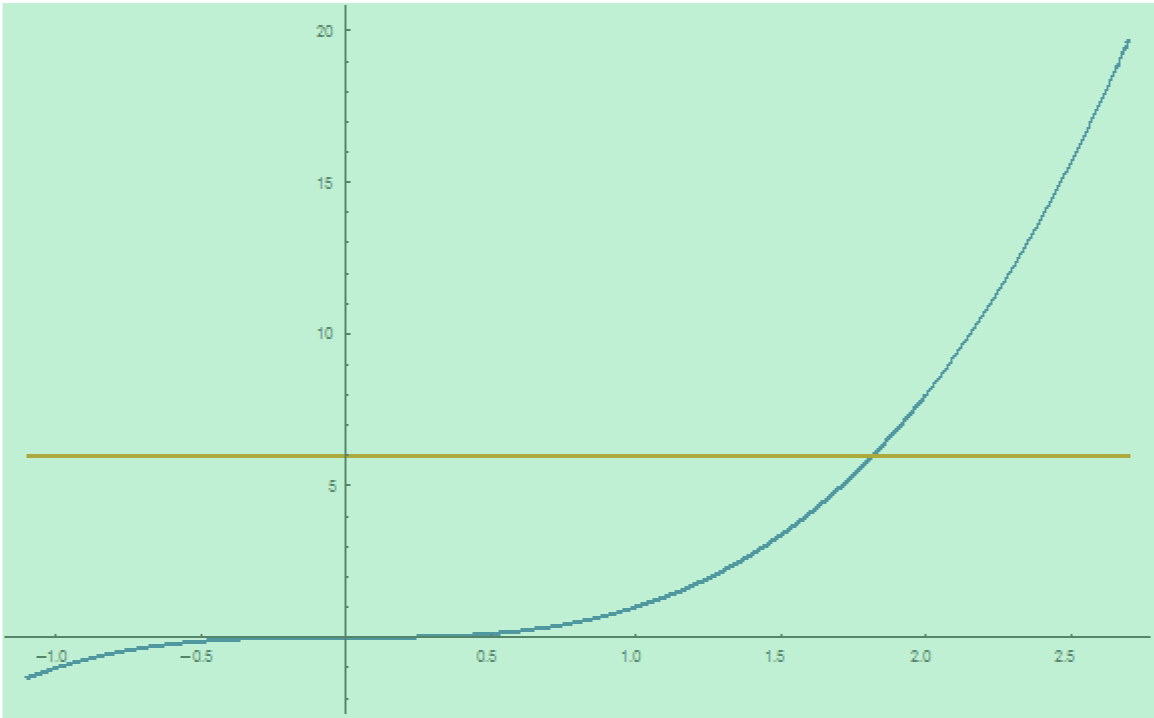

Figure 1. The graph of $y = x^3$ and its 3rd derivative are pictured, color coded accordingly.

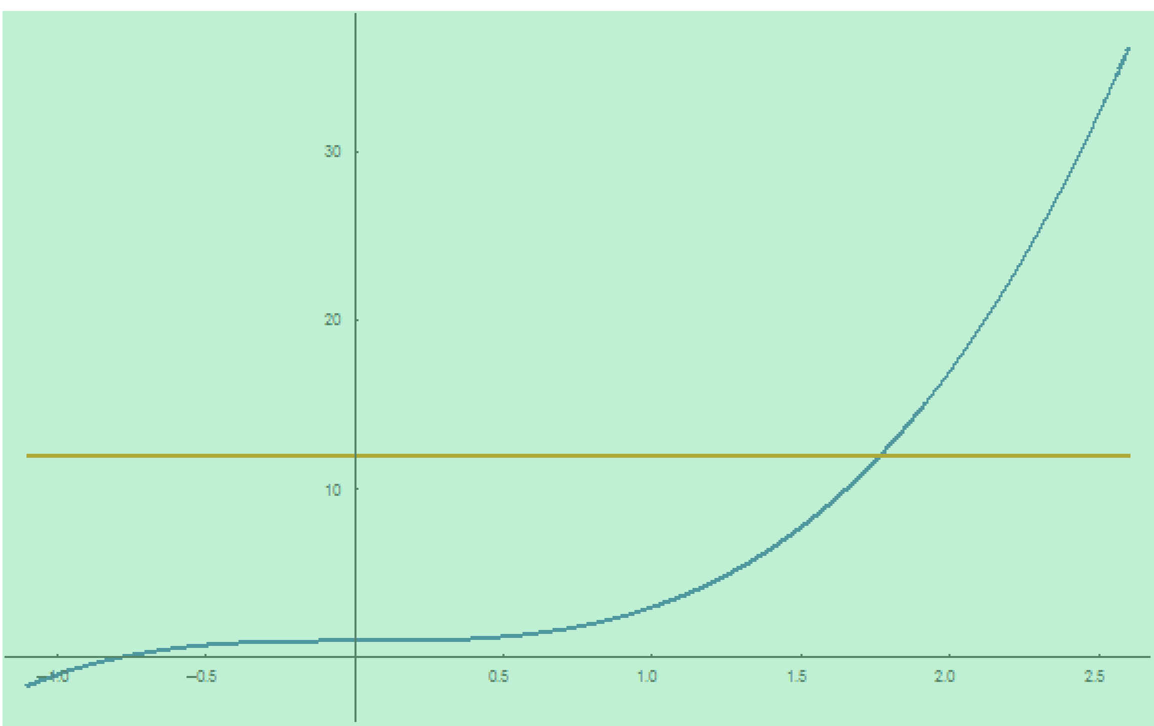

Figure 2. The graph of $y = 2x^3 + 1$ and its 3rd derivative are pictured, color coded accordingly.

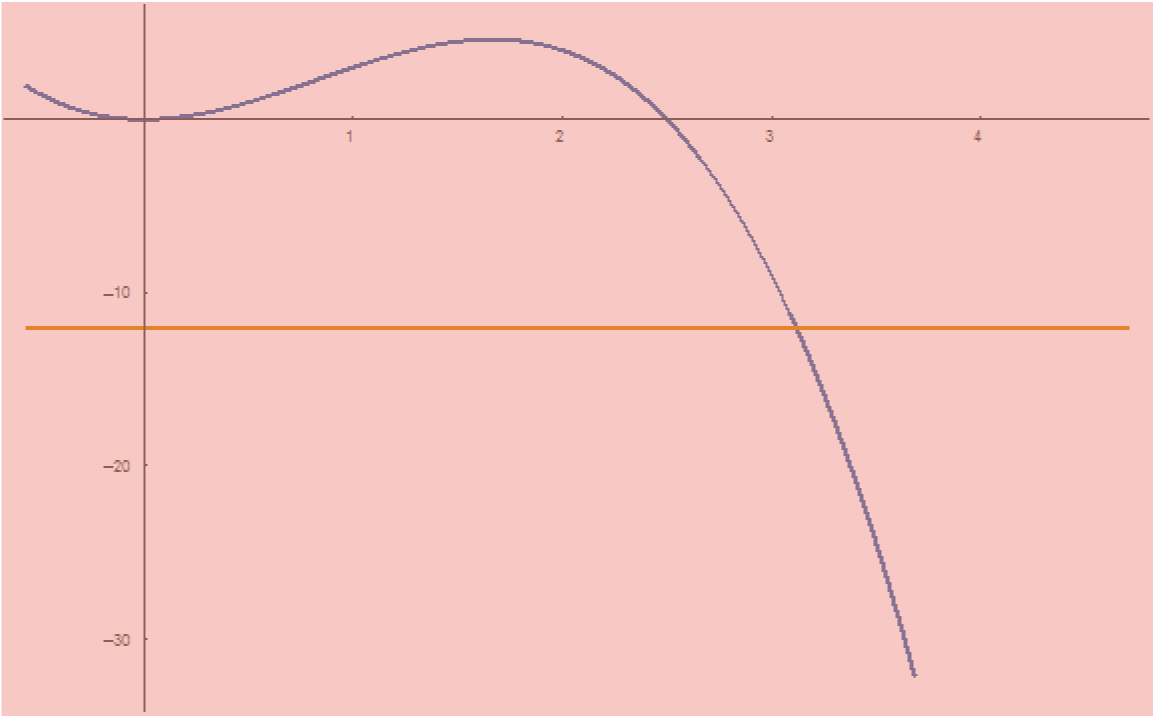

Figure 3. The graph of $y = -2x^3 + 5x^2$ and its 3rd derivative are pictured, color coded accordingly.

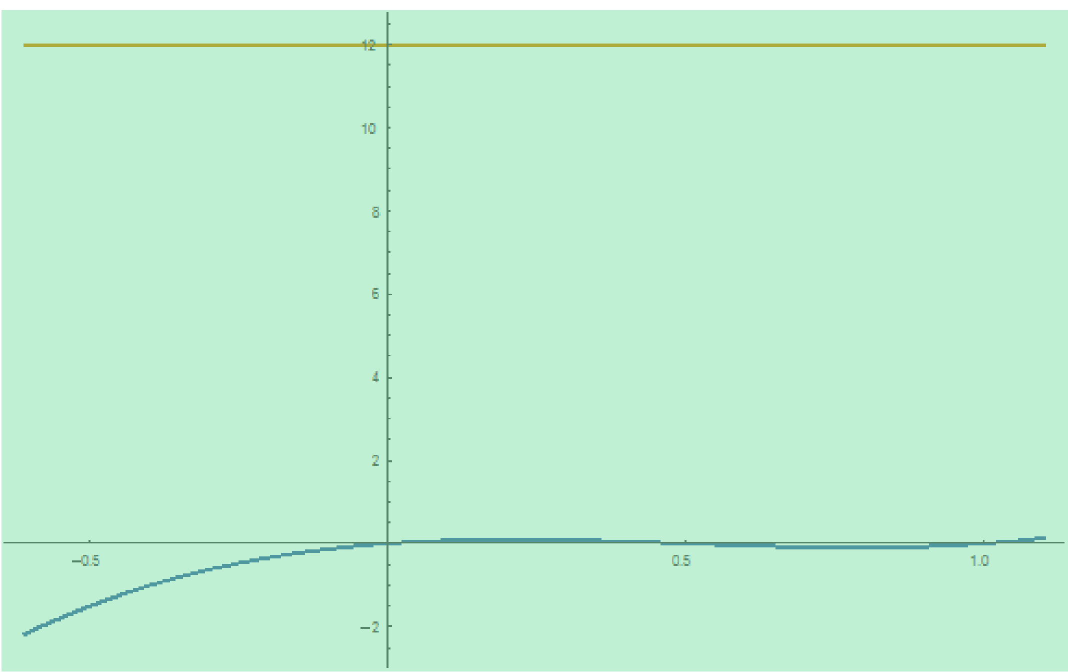

Figure 4. The graph of $y = 2x^3 - 3x^2 + x$ and its 3rd derivative are pictured, color coded accordingly.

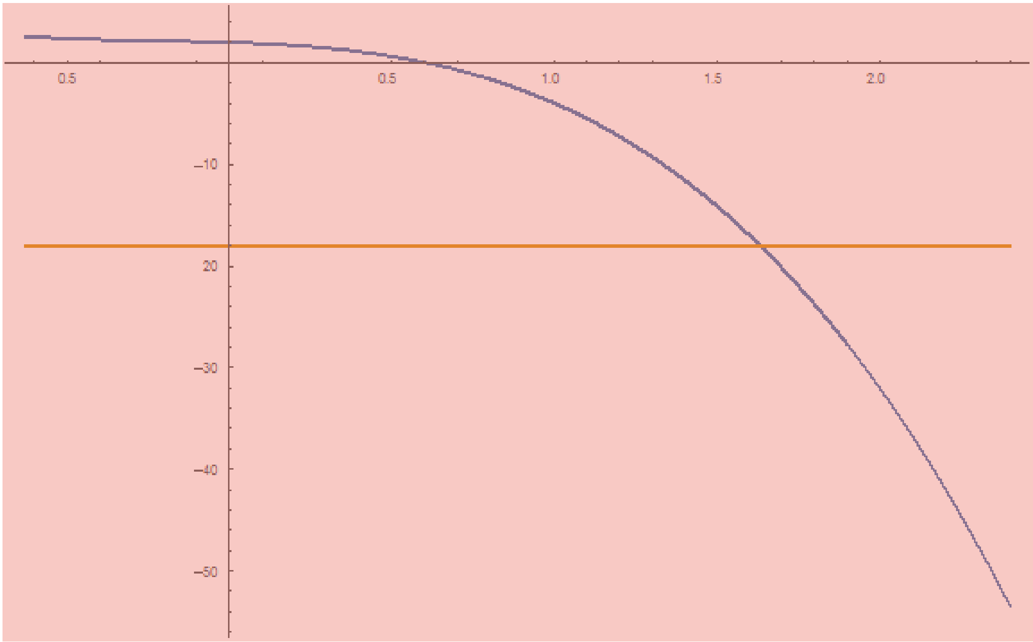

Figure 5. The graph of $y = -3x^3 - 2x^2 - x + 2$ and its 3rd derivative are pictured, color coded accordingly.

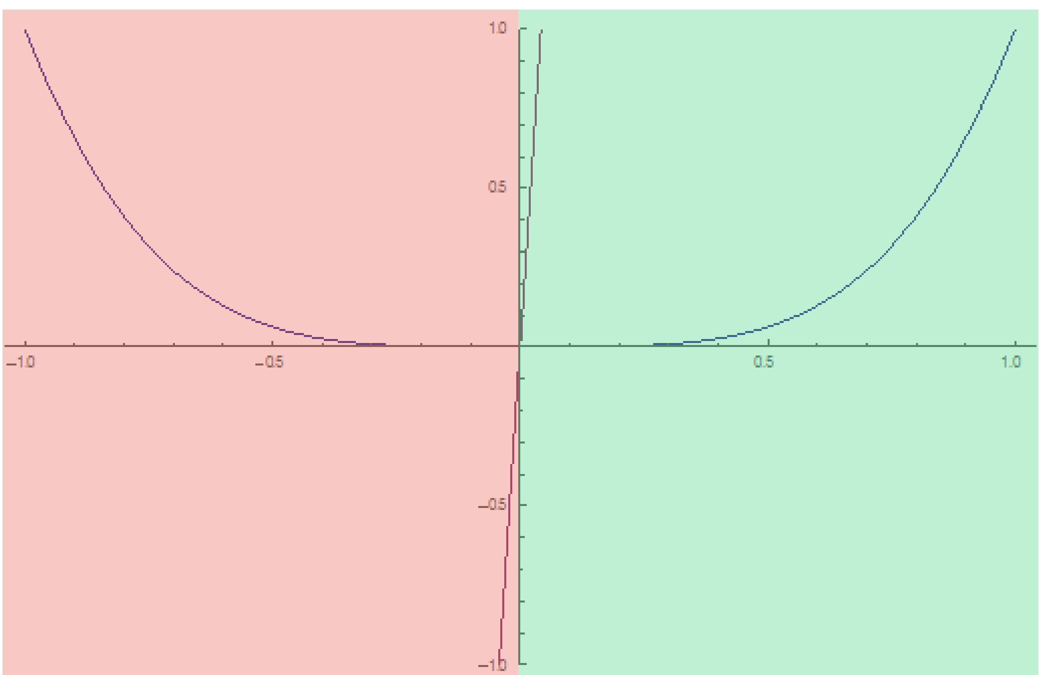

Figure 6. The graph of $y = x^4$ and its 3rd derivative are pictured, color coded accordingly. It should be noted that the boundaries of this graph were manually adjusted to x{-1, 1} y{-1, 1} as the defaults provided unintelligible results.

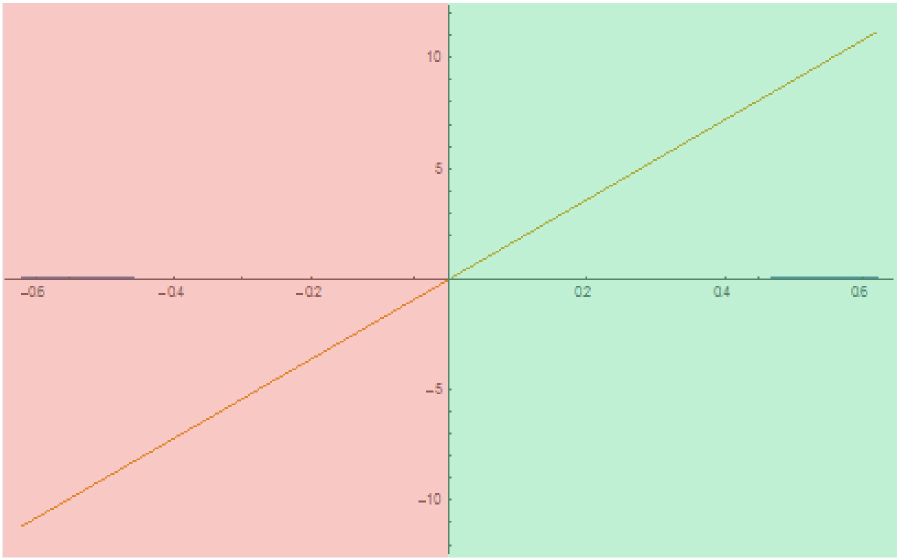

Figure 7. The graph of $y = \frac{3}{4}x^4$ and its 3rd derivative are pictured, color coded accordingly. Similar to Figure 6, the default boundaries provided unhelpful results. However, due to the similarities between this equation and the equation used in Figure 6, the defaults were kept for this figure in the interest of thoroughness.

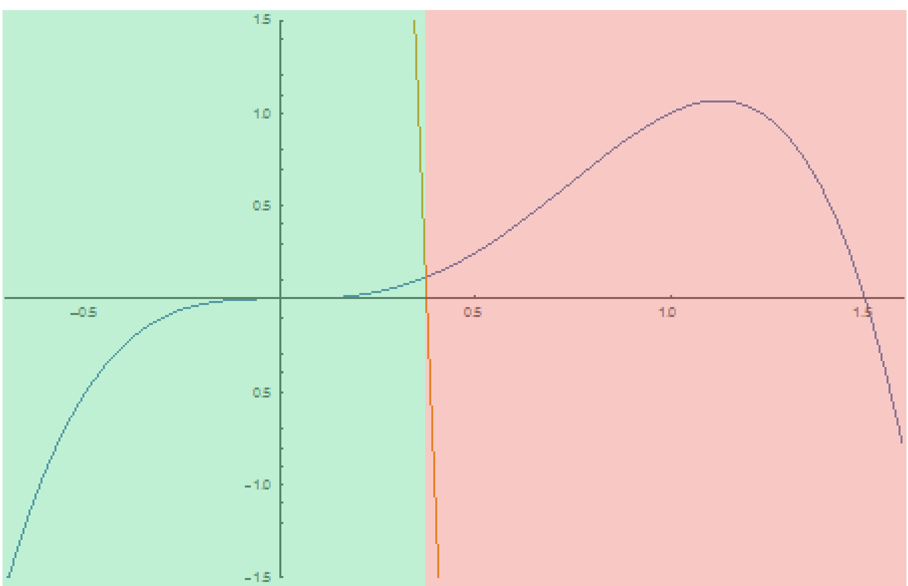

Figure 8. The graph of $y = -2x^4 + 3x^3$ and its 3rd derivative are pictured, color coded accordingly. Another instance where it should be noted that the boundaries were manually adjusted.

Figure 9. The graph of $y = 2x^4 - \frac{3}{2}x^3 + 3x^2$ and its 3rd derivative are pictured, color coded accordingly. Another instance where it should be noted that the boundaries were manually adjusted.

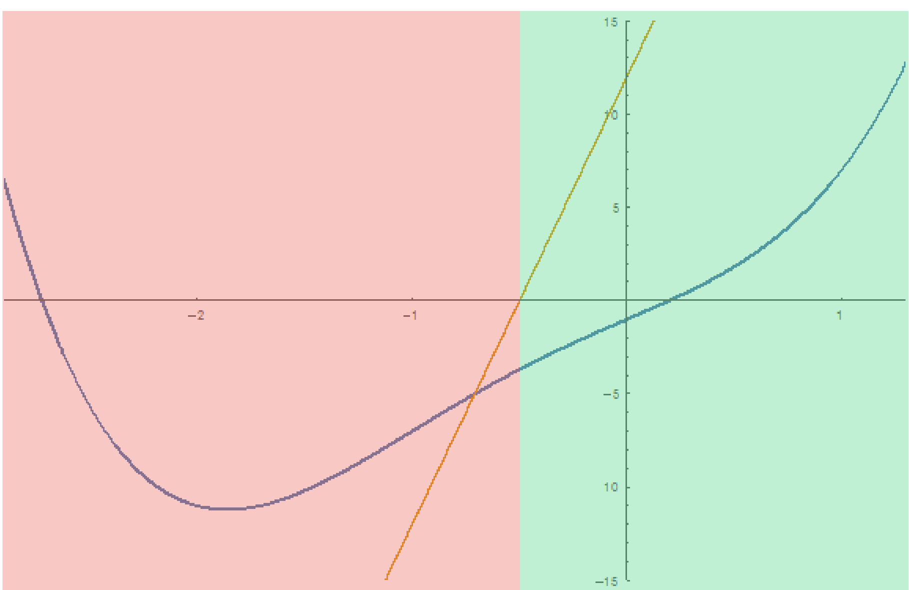

Figure 10. The graph of $y = x^4 + 2x^3 + 5x - 1$ and its 3rd derivative are pictured, color coded accordingly.

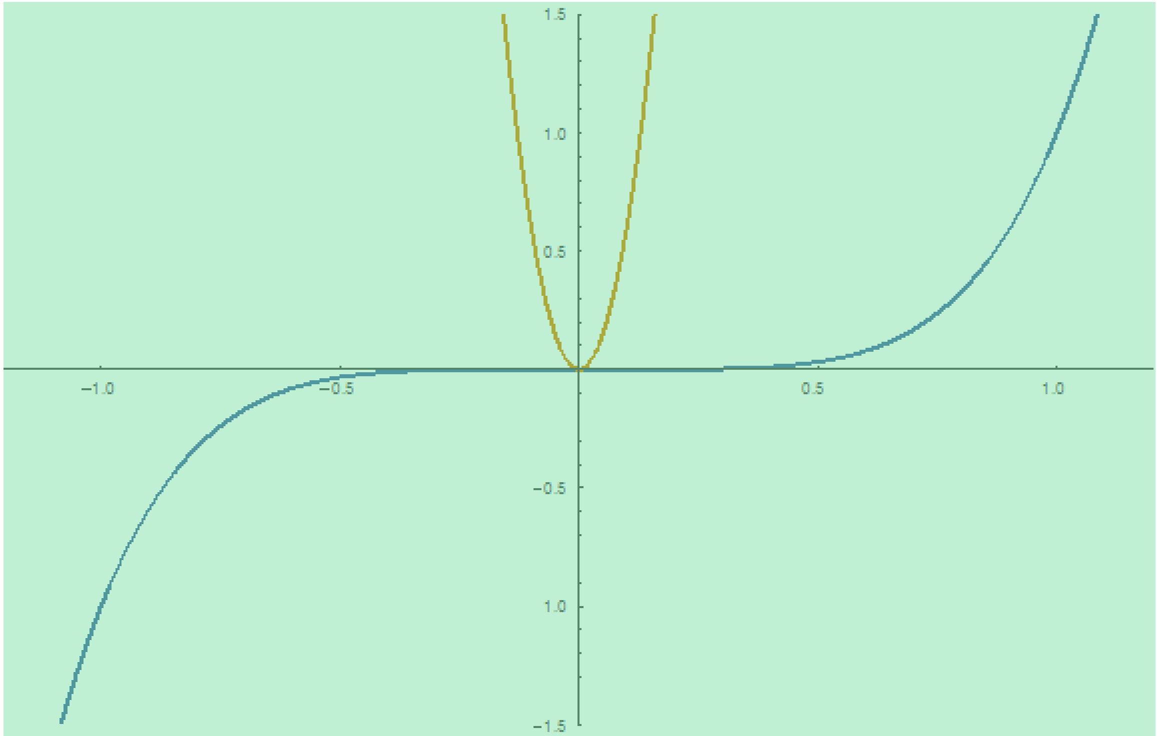

Figure 11. The graph of $y = x^5$ and its 3rd derivative are pictured, color coded accordingly.

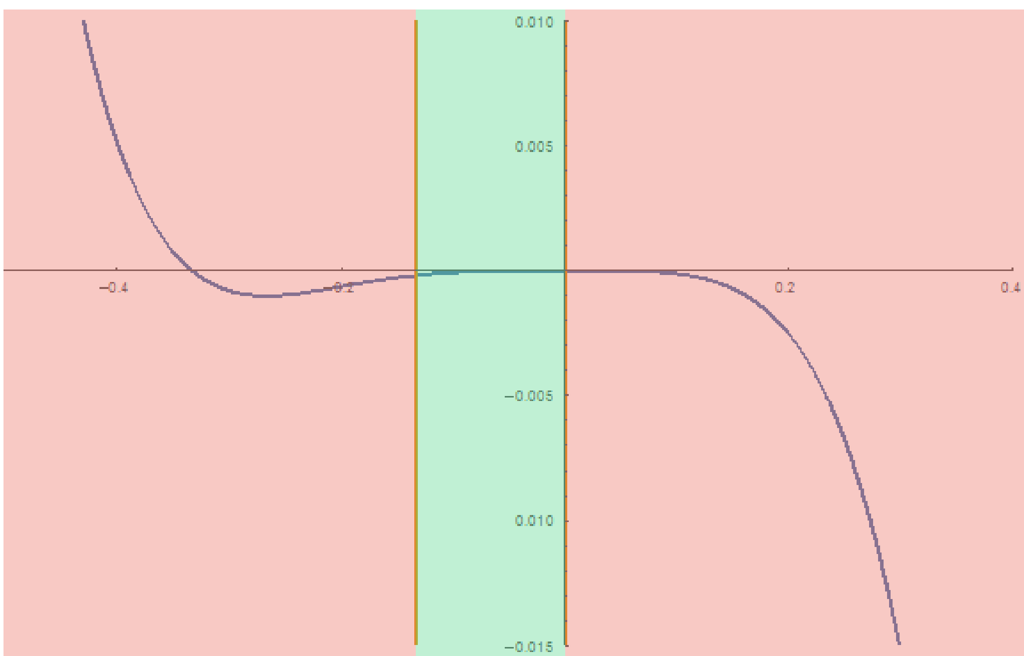

Figure 12. The graph of $y = -3x^5 - x^4$ and its 3rd derivative are pictured, color coded accordingly. Another instance where it should be noted that the boundaries were manually adjusted.

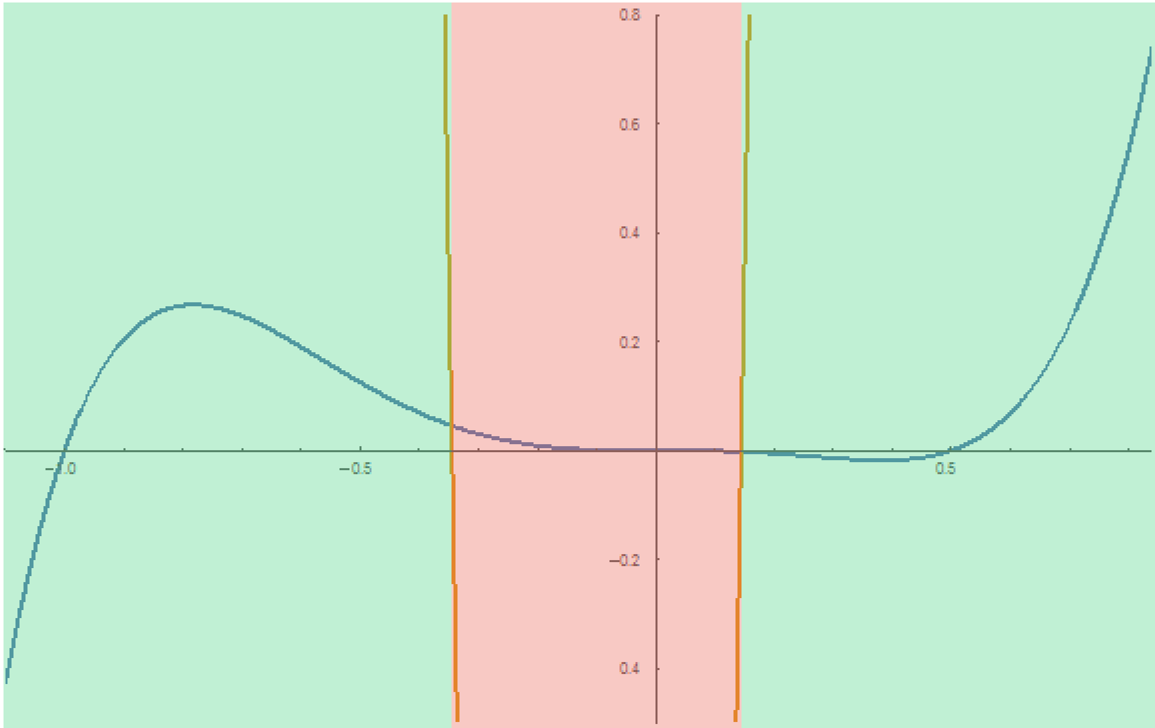

Figure 13. The graph of $y = 2x^5 + x^4 - x^3$ and its 3rd derivative are pictured, color coded accordingly.

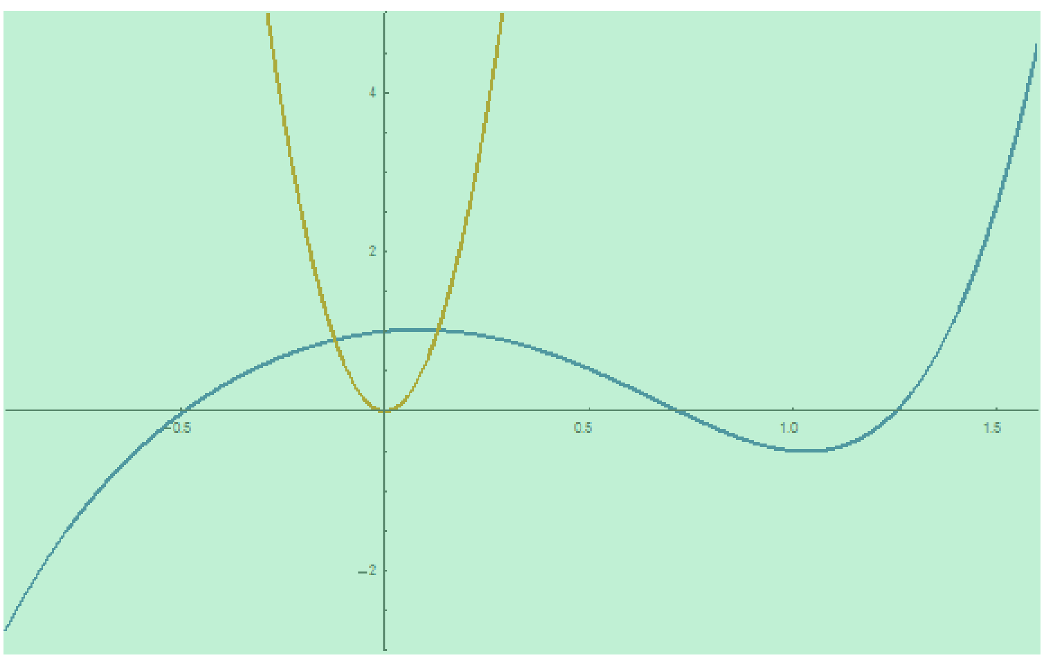

Figure 14. The graph of $y = x^5 - 3x^2 + \frac{1}{2}x + 1$ and its 3rd derivative are pictured, color coded accordingly.

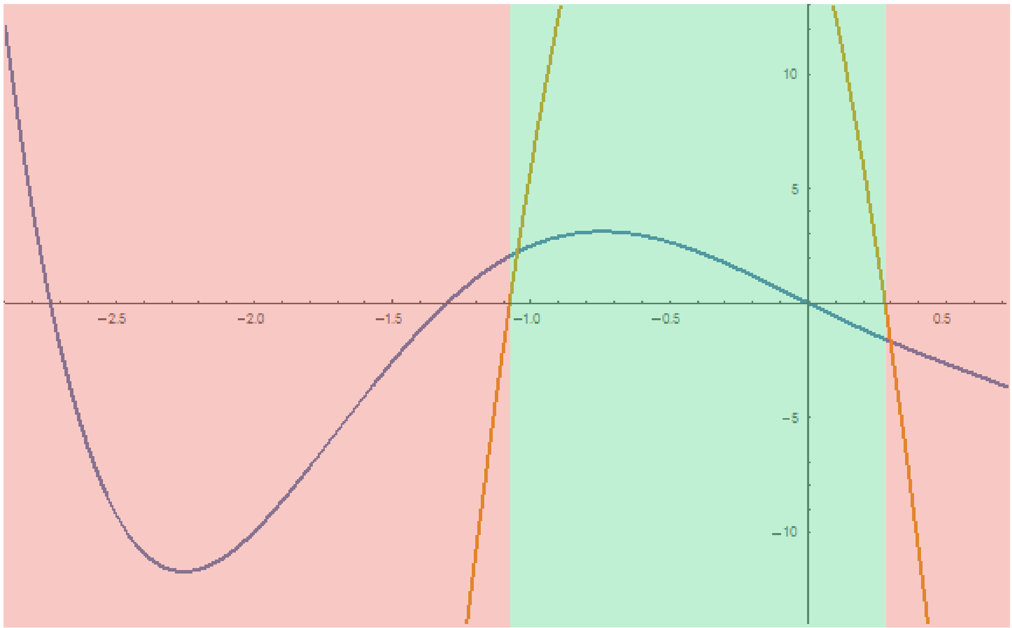

Figure 15. The graph of $y = -x^5 - 2x^4 + 3x^3 + \frac{1}{2}x^2 - 6x$ and its 3rd derivative are pictured, color coded accordingly.

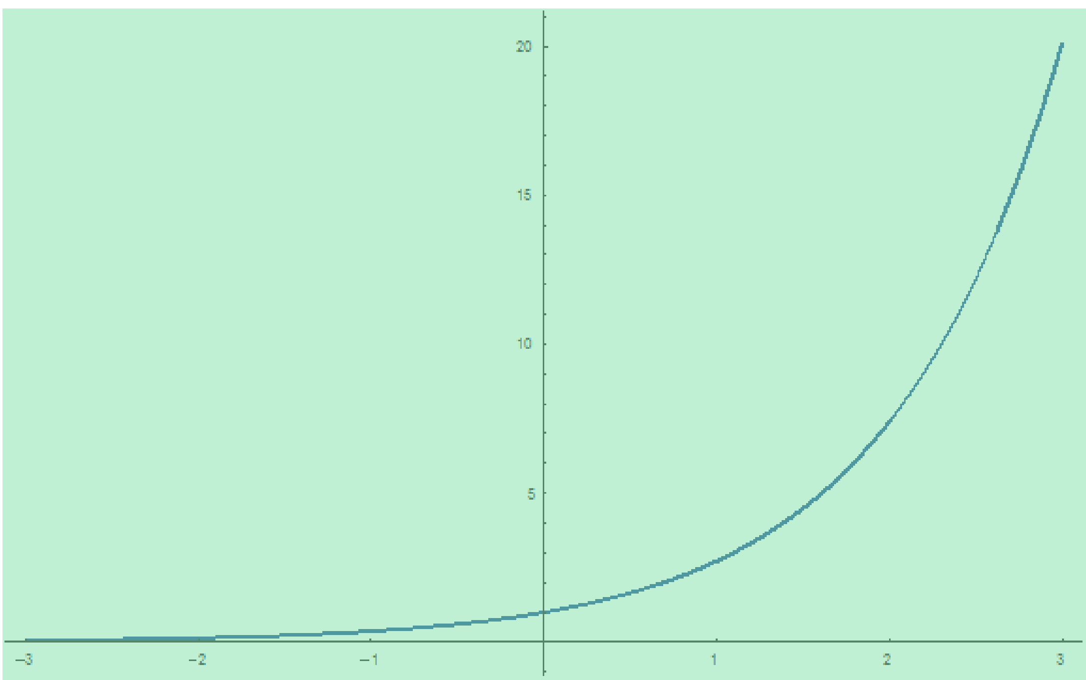

Figure 16. The graph of $y = e^x$ and its 3rd derivative are pictured, color coded accordingly. It should be noted that only the original function was graphed, as the 3rd derivative was identical to the original function.

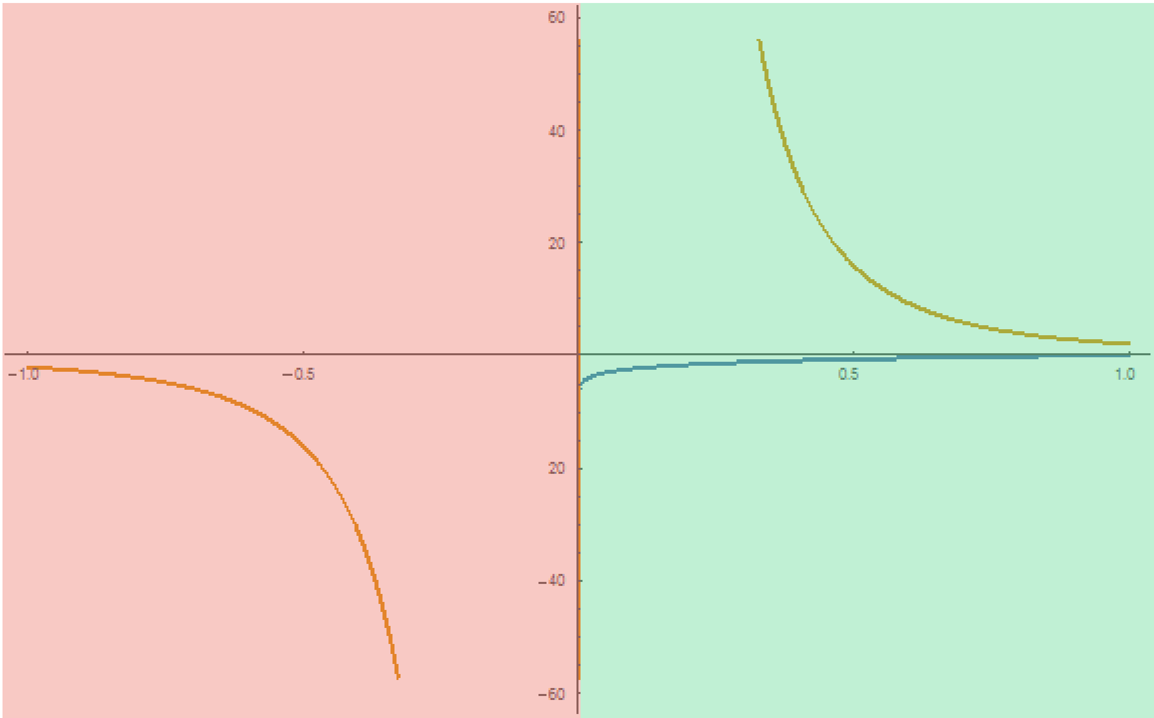

Figure 17. The graph of $y = \ln(x)$ and its 3rd derivative are pictured, color coded accordingly.

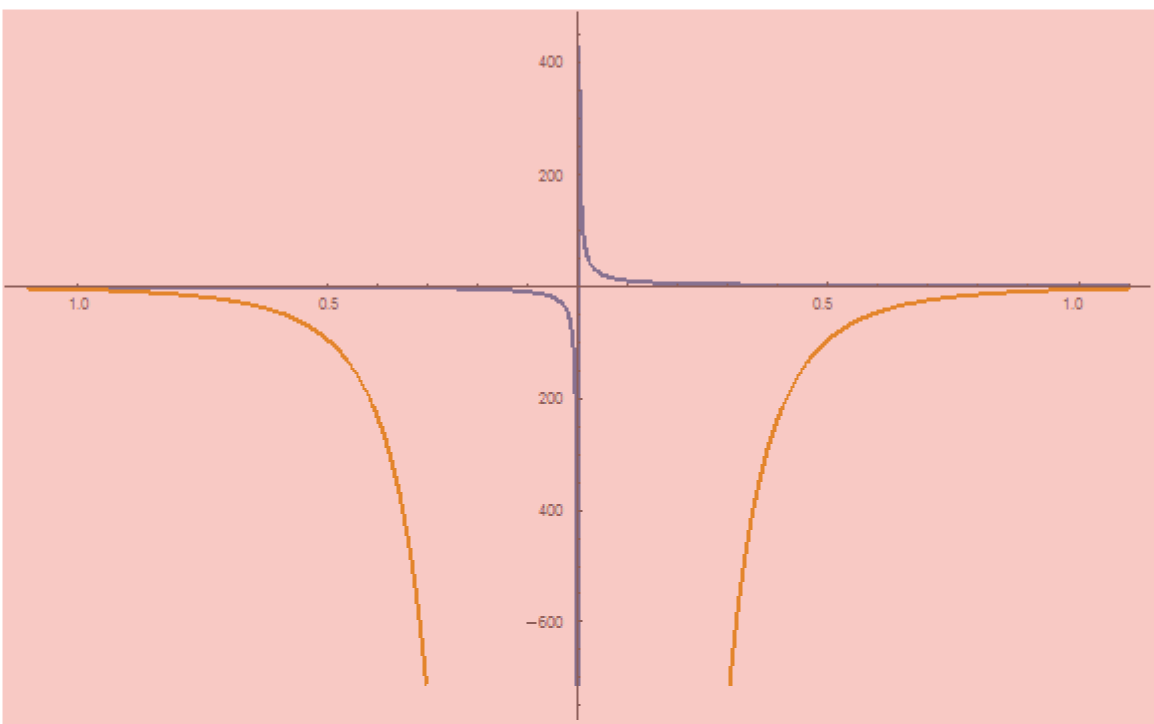

Figure 18. The graph of $y = \frac{1}{x}$ and its 3rd derivative are pictured, color coded accordingly.

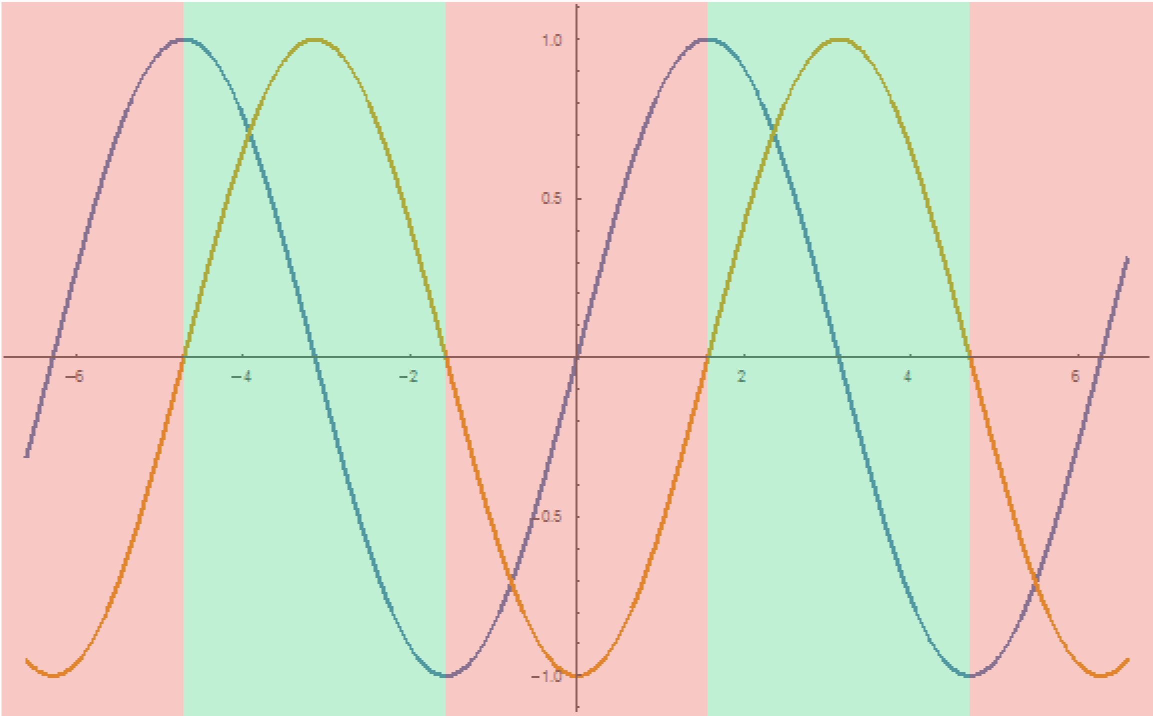

Figure 19. The graph of $y = \sin(x)$ and its 3rd derivative are pictured, color coded accordingly.

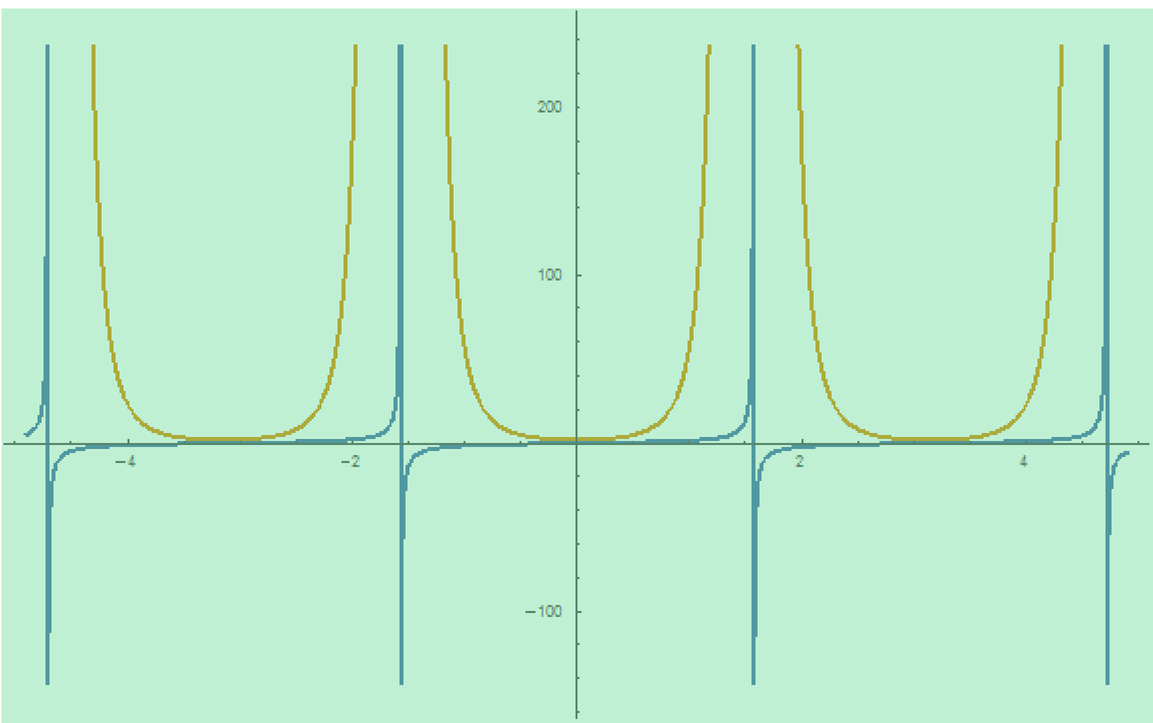

Figure 20. The graph of $y = \tan(x)$ and its 3rd derivative are pictured, color coded accordingly.

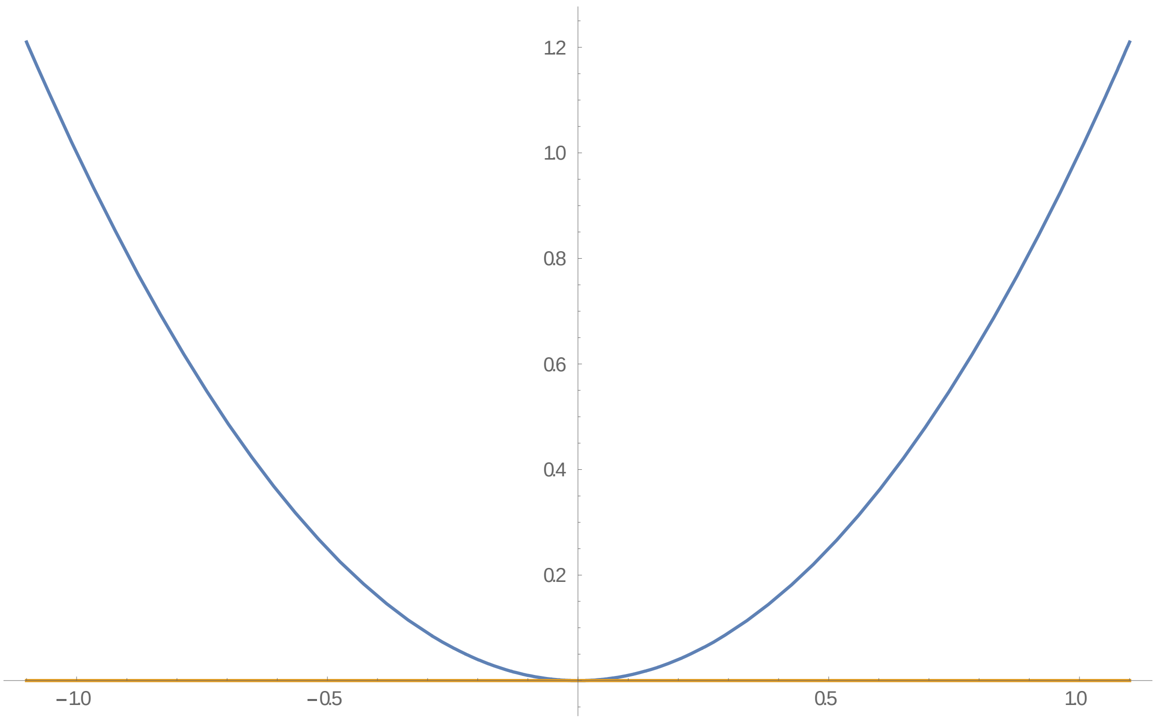

Figure 21. The graph of $y = x^2$ and its 3rd derivative are pictured. It should be noted that the third derivative is 0 and, thus, no coloring is present.

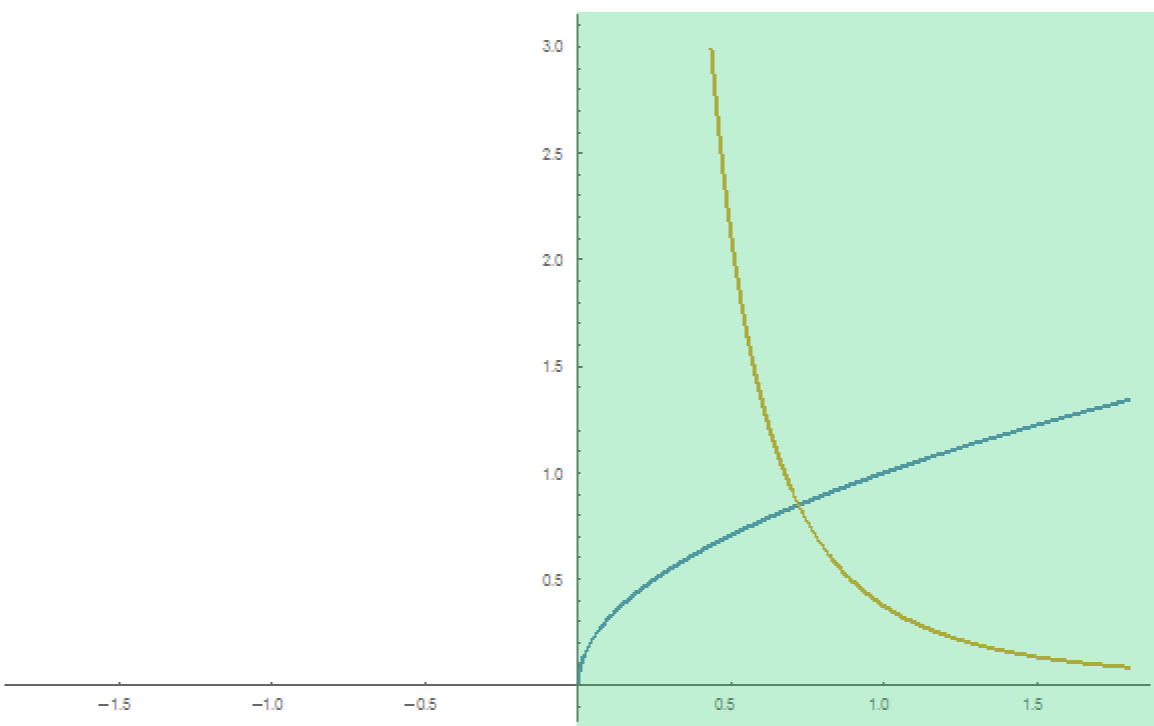

Figure 22. The graph of $y = \sqrt{x}$ and its 3rd derivative are pictured, color coded accordingly.

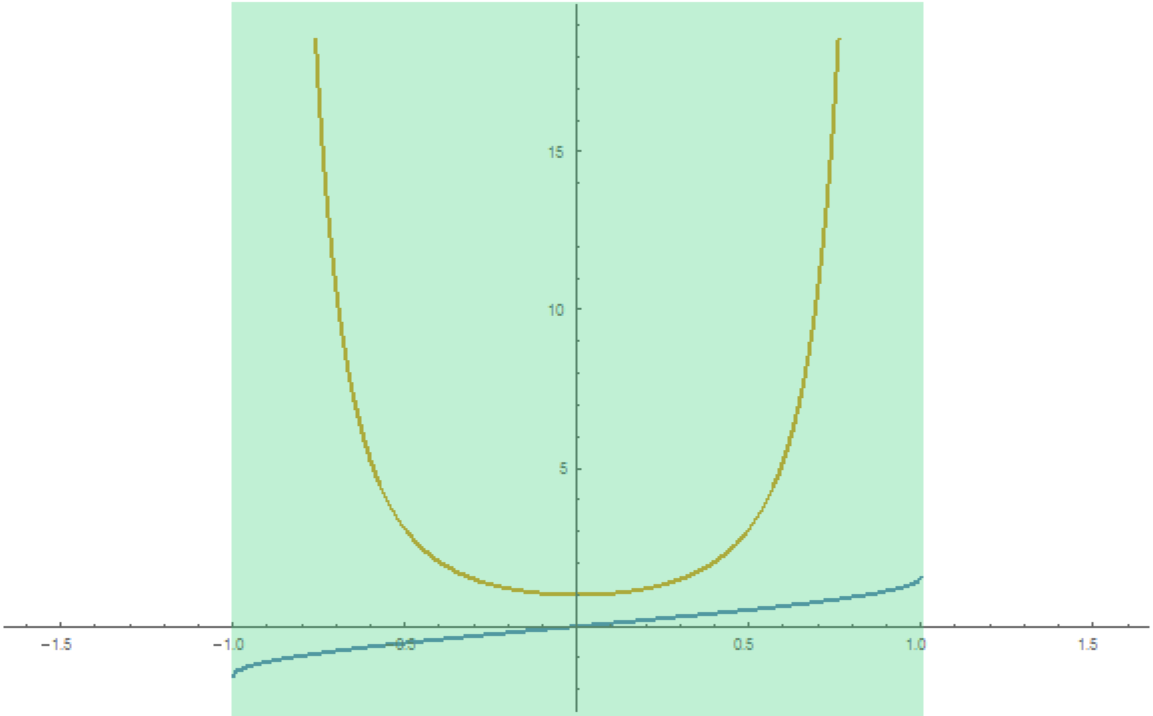

Figure 23. The graph of $y = \arcsin(x)$ and its 3rd derivative are pictured, color coded accordingly.

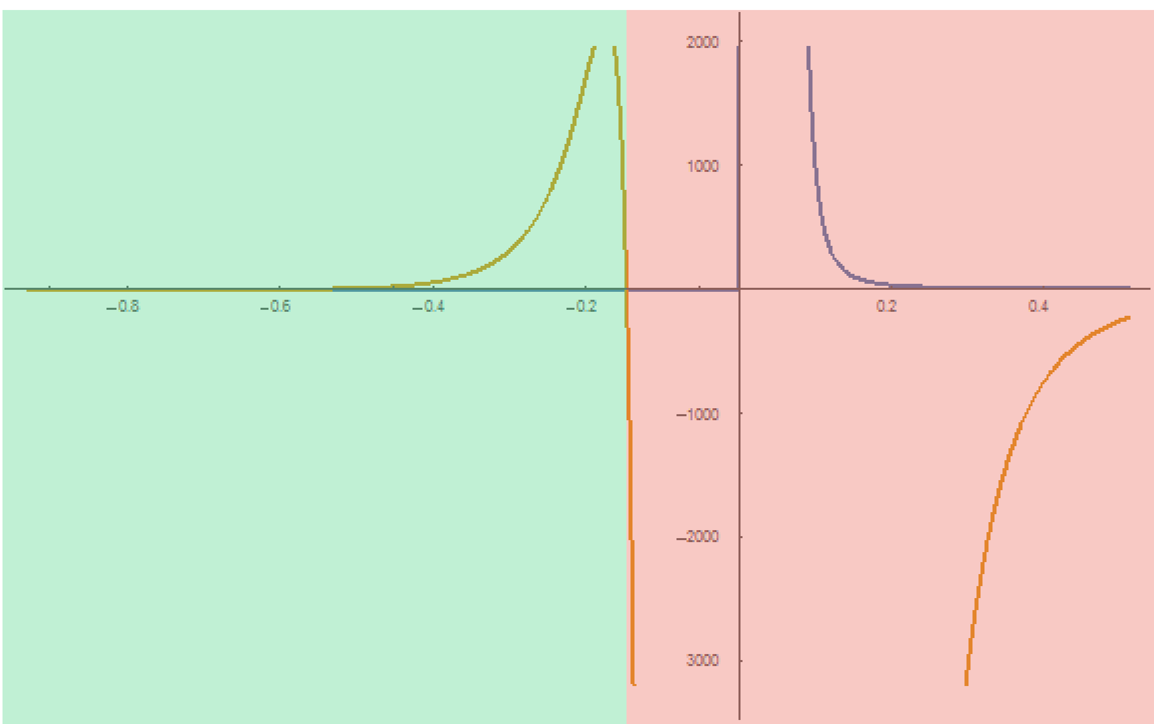

Figure 24. The graph of $y = 2^{\frac{1}{x}}$ and its 3rd derivative are pictured, color coded accordingly.

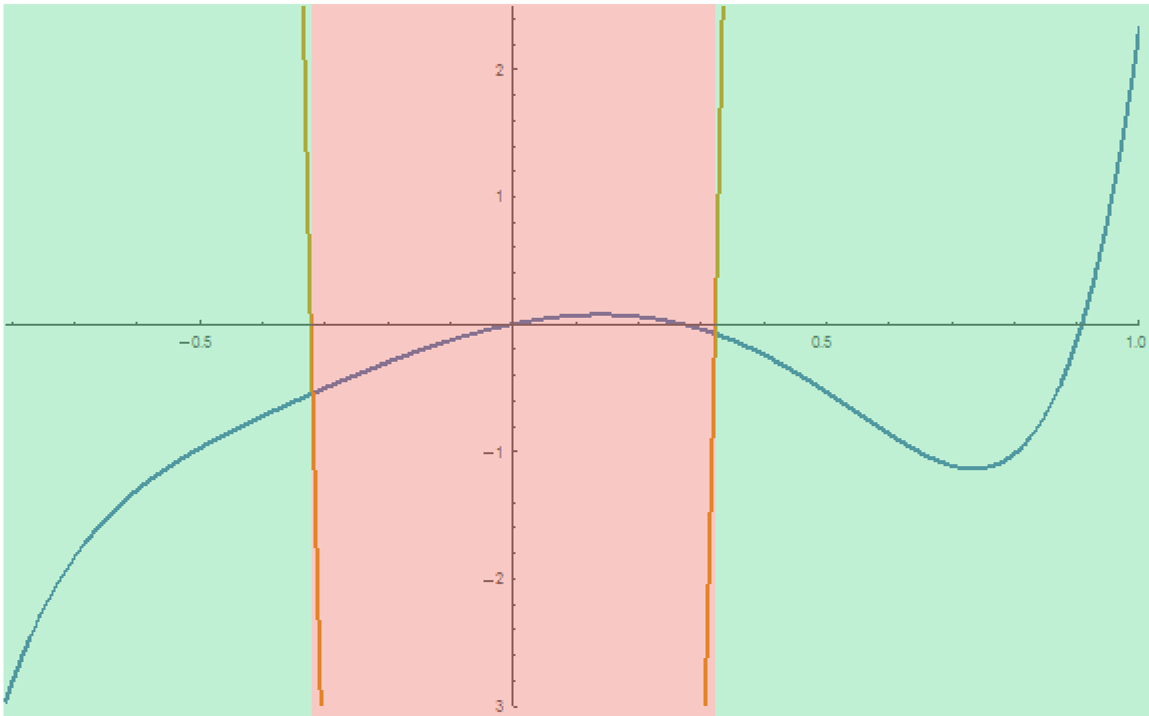

Figure 25. The graph of $y = 7x^7 - \frac{8}{3}x^3 - 3x^2 + x$ and its 3rd derivative are pictured, color coded accordingly.